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Electrodynamics of moving media and the Čerenkov radiation

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Abstract. The object of this paper is to find the equations for the electric and magnetic intensities **E** and **H** in a homogeneous medium moving with a uniform velocity, from Maxwell's field equations and Minkowski's material relations. The nature of the solution of these equations in general is studied. An alternative approach is suggested, to reduce the problem to a mathematically equivalent problem in empty space. This is done by introducing linear space–time transformations similar to the Lorentz transformations. As an application, the Čerenkov radiation is studied. By suitable transformation, it is reduced to a problem equivalent to antenna radiation. The special case of motion with velocity equal to the phase velocity, which has unusual consequences, is included in an appendix.

1. Introduction

In recent years there has been a revival of interest in the studies of the electromagnetic properties of moving media. The investigations may be broadly classified into two groups. The main interest of one group is to regard it as a general problem in the classical theory of fields and to find its relation to the theory of relativity, both special and general. The object of the other group is mostly connected with practical applications, such as its relation to Čerenkov radiation, radiations from high-energy plasma, etc. Our studies are confined to the second. By Čerenkov radiation we mean the radiation emitted by charges and currents which are space–time functions of the form $f(\mathbf{r}-\mathbf{v}t)$, where $|\mathbf{v}|$ is greater than u, the phase velocity of electromagnetic waves in the medium. Usually Čerenkov radiation is attributed to radiation due to charges moving with constant velocity in this range, i.e. $f(\mathbf{r}-\mathbf{v}t) \propto \delta(\mathbf{r}-\mathbf{v}t)$, with $|\mathbf{v}| = v > u$. The nature of the radiation emitted by a particle moving with varying velocity such that $v > u$ has been studied by the author (Sen Gupta 1965). In this range, though the charge current may be made static (or purely accelerating in the case of varying velocity) in a suitable Lorentz frame, the characteristic properties of Maxwell's field equations change with respect to space and time. This is due to the involved nature of Minkowski's material equations. The object of this paper is to examine critically the appearance of the Čerenkov radiation in relation to the change of properties of the field equations in a material medium. Since this change is due to the material equation, the radiation is basically a property of the medium. Hence an adequate theory of the Čerenkov radiation should be microscopic in nature. Although a satisfactory microscopic theory of the electromagnetic properties of a material medium is yet to be formulated, nevertheless, a good beginning for a microscopic theory of the Čerenkov radiation has been made by Pratap (1967). With these limitations it is still instructive to study the macroscopic equations and to examine their consequences.

With this aim, we have tried to investigate the nature of the field equations in a homogeneous moving medium in general. An attempt to study the Čerenkov radiation by this method has been made by Nag and Sayied (1956). Their special interest has limited their discussions, and the general character of the equations and the field quantities have not been revealed. Most of the authors who have investigated the problem have used vector and scalar potentials to describe the field. The nature of Minkowski's material equations makes the equations for the potential involved to such an extent that they cannot be suitably simplified, even with a freedom of choice of gauge. On the other hand, as our procedure shows, the field equations directly in terms of **E** and **H** are not unwieldy, though by no means simple. The Čerenkov-like radiations in a plasma studied by Majumdar (1960, 1961) are distinct from the Čerenkov radiation, as envisaged in this paper.

Further, as a continuation of our objective to see the difference between the two cases $v > u$ and $v < u$, we have reduced the problem similar to that in empty space by introducing linear space–time transformations which are analogous to the Lorentz transformation; for $v < u$ this is the same as the Lorentz transformation with c replaced by u, and for $v > u$ with c replaced by u and v replaced by u^2/v. With v constant, this reduces in this first case to a static one as the convective current diappears. In the other case the charge disappears, and hence it reduces to an antenna problem. This clearly demonstrates the radiative property in the latter case.

In the next section the field equations for **E** and **H** are obtained by eliminating **B** and **D** from Maxwell's equations and Minkowski's equations. The nature of their solutions and the corresponding Poynting theorem are discussed. Section 3 is devoted to an alternative approach, namely the transformation of the field equation to a static one and to that of an antenna by suitable linear transformation of space and time. The Čerenkov radiation is studied as an application in § 4. The paper is supplemented with an appendix in which the special case of $v = u$ is discussed; this has the consequence that **E** and **H** depend only on the current and not on the charge

2. The field equations

Maxwell's equations for electromagnetic fields in a homogeneous material medium are given by

$$\nabla \wedge \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \tag{1}$$

$$\nabla \wedge \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \tag{2}$$

$$\nabla . \mathbf{D} = P \tag{3}$$

and

$$\nabla . \mathbf{B} = 0. \tag{4}$$

For our discussions we take the medium to be non-dispersive. However, most of the results obtained may be extended to dispersive media. The material equations of Minkowski for a homogeneous medium moving with uniform velocity **v** with respect to an observer are

$$\mathbf{D} + \beta\mathbf{n} \wedge \mathbf{H} = \epsilon(\mathbf{E} + \beta\mathbf{n} \wedge \mathbf{B}) \tag{5}$$

$$\mathbf{B} - \beta\mathbf{n} \wedge \mathbf{E} = \mu(\mathbf{H} - \beta\mathbf{n} \wedge \mathbf{D}) \tag{6}$$

where $\mathbf{n} = \mathbf{v}/v$ and $\beta = v/c$. Thus

$$\mathbf{D}(1-\epsilon\mu\beta^2) = \epsilon(1-\beta^2)\mathbf{E} + \beta(\epsilon\mu-1)\{\mathbf{n} \wedge \mathbf{H} - \epsilon\beta\mathbf{n}(\mathbf{n} . \mathbf{E})\} \tag{7}$$

$$\mathbf{B}(1-\epsilon\mu\beta^2) = \mu(1-\beta^2)\mathbf{H} - \beta(\epsilon\mu-1)\{\mathbf{n} \wedge \mathbf{E} + \mu\beta\mathbf{n}(\mathbf{n} . \mathbf{H})\}. \tag{8}$$

It is to be noted that for the parallel components

$$\mathbf{n} . \mathbf{D} = \epsilon(\mathbf{n} . \mathbf{E}) \tag{7'}$$

$$\mathbf{n} . \mathbf{B} = \mu(\mathbf{n} . \mathbf{H}) \tag{8'}$$

i.e. they are the same as those when the medium is at rest. Hence, excluding the special case $1-\epsilon\mu\beta^2 = 0$, i.e. the phase velocity of the electromagnetic wave in the medium is equal to v, one can find **D** and **B** from equations (7) and (8) in terms of **E** and **H**†. The case of $v = u$ requires special attention: it is discussed in the appendix. In order to write

† There is a typographical slip in Pauli's (1921) article in the sign of the last term of the right-hand side of equation (8) with coefficient μ. This has been reproduced in the English translation, and carried over in many other books.

Maxwell's equations in terms of **E** and **H**, it is convenient to introduce the notation

$$\bar{\nabla} = \nabla + \mathbf{n}\frac{1-\xi}{v}\frac{\partial}{\partial t} \tag{9}$$

$$\bar{\mathbf{E}} = \mathbf{E} + \mathbf{n}(\mathbf{n}\,.\,\mathbf{E})\left(\frac{1}{\xi} - 1\right) \tag{10}$$

and

$$\bar{\mathbf{H}} = \mathbf{H} + \mathbf{n}(\mathbf{n}\,.\,\mathbf{H})\left(\frac{1}{\xi} - 1\right) \tag{11}$$

where

$$\xi = \frac{1-\beta^2}{1-\epsilon\mu\beta^2}. \tag{12}$$

ξ increases from 1 to ∞ as v changes from 0 to u, and ξ is negative when $v > u$. **E** and $\bar{\mathbf{E}}$ differ only along **n** and $\mathbf{E}.\mathbf{n} = \xi(\bar{\mathbf{E}}.\mathbf{n})$; similarly for **H**. With this notation it is easy to see that equations (1)–(4) can be written in the form

$$\bar{\nabla} \wedge \mathbf{H} - \frac{\epsilon}{c}\xi\frac{\partial \bar{\mathbf{E}}}{\partial t} = \mathbf{J} \tag{13}$$

$$\bar{\nabla} \wedge \mathbf{E} + \frac{\mu}{c}\xi\frac{\partial \bar{\mathbf{H}}}{\partial t} = 0 \tag{14}$$

$$\bar{\nabla}\,.\,\bar{\mathbf{E}} = \frac{1}{\epsilon\xi}\bar{P} \tag{15}$$

and

$$\bar{\nabla}\,.\,\bar{\mathbf{H}} = 0 \tag{16}$$

where

$$\bar{P} = P + \frac{\xi-1}{\beta}(\mathbf{n}.\mathbf{J}). \tag{17}$$

In the above equations both **E**, **H** and $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ appear, which makes further elimination a little involved. However, the components of equations (13) and (14) along **n** remain unchanged; they are

$$\mathbf{n}\,.\,\nabla \wedge \mathbf{E} + \frac{\mu}{c}\frac{\partial}{\partial t}(\mathbf{n}\,.\,\mathbf{H}) = 0 \tag{18}$$

$$\mathbf{n}\,.\,\nabla \wedge \mathbf{H} - \frac{\epsilon}{c}\frac{\partial}{\partial t}(\mathbf{n}\,.\,\mathbf{E}) = \mathbf{n}\,.\,\mathbf{J}. \tag{19}$$

This follows directly from equations (1), (2), (7′) and (8′). From equations (13)–(16) it is clear that the components of $\bar{\mathbf{E}}$, **H** no longer satisfy the same type of wave equation; hence the simultaneous equations cannot be reduced easily to independent equations. By forming the scalar product of equations (13) and (14) with $\mathbf{n} \wedge \nabla$ and using equations (7′), (8′) and (15)–(19), one can find the wave equations for $\mathbf{n}.\mathbf{E}$ and $\mathbf{n}.\mathbf{H}$ as follows:

$$\mathscr{D}(\mathbf{n}\,.\,\mathbf{E}) = \frac{1}{\epsilon\xi}\left\{(\mathbf{n}\,.\,\bar{\nabla})\bar{P} + \frac{\xi^2 c}{u^2}\frac{\partial}{\partial t}(\mathbf{n}.\mathbf{J})\right\} \tag{20}$$

$$\mathscr{D}(\mathbf{n}\,.\,\mathbf{H}) = -\mathbf{n} \wedge \nabla\,.\,\mathbf{J} \tag{21}$$

where

$$\mathscr{D} \equiv \mathbf{n} \wedge \nabla\,.\,\mathbf{n} \wedge \nabla + \frac{1}{\xi}(\mathbf{n}\,.\,\bar{\nabla})^2 - \frac{\xi}{u^2}\frac{\partial^2}{\partial t^2}. \tag{22}$$

The equations for the other components of **E** and **H** are obtained from equations (13) and (14) by forming the vector product with $\mathbf{n} \wedge \nabla$:

$$\left\{(\mathbf{n}\,.\,\bar{\nabla})^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2}\right\}\mathbf{n} \wedge \mathbf{H} = \mathbf{n} \wedge \left[\nabla(\mathbf{n}\,.\,\bar{\nabla})(\mathbf{n}\,.\,\mathbf{H}) - \mathbf{n} \wedge \left\{\frac{\epsilon\xi}{c}\nabla\frac{\partial}{\partial t}(\mathbf{n}\,.\,\mathbf{E}) + (\mathbf{n}\,.\,\bar{\nabla})\mathbf{J}\right\}\right] \tag{23}$$

$$\left\{(\mathbf{n}\,.\,\bar{\nabla})^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2}\right\}\mathbf{n} \wedge \mathbf{E} = \mathbf{n} \wedge \left[\nabla(\mathbf{n}\,.\,\bar{\nabla})(\mathbf{n}\,.\,\mathbf{E}) + \frac{\mu\xi}{c}\frac{\partial}{\partial t}\left\{\mathbf{n} \wedge \nabla(\mathbf{n}.\mathbf{H}) - \mathbf{J}\right\}\right]. \tag{24}$$

The above equations may be solved after determining $\mathbf{n}.\mathbf{H}$ and $\mathbf{n}.\mathbf{E}$ from equations (21) and (22). The equations (20), (21) and (23), (24) are similar to those in the problems of wave-guides where they are simply algebraic equations. There it is because of the special nature of the solutions, which are of the form $\exp[i\{k(\mathbf{n}\,.\,\mathbf{r}) - \omega t\}]$. It should be mentioned that the Maxwell equations in a rest system may also be separated in this manner, but they will introduce unnecessary complications; hence they are redundant owing to the homogeneity of space. But, owing to the motion of the medium, the space is no longer isotropic and such a separation of the field equation is imperative here. Since the differential operator on the left-hand side of equations (23) and (24) does not contain $\mathbf{n} \wedge \nabla$, the dependence of the components of **E** and **H** perpendicular to **n**, on $\mathbf{n} \wedge \mathbf{r}$, is determined directly by the components along **n** and (P, **J**).

Next, in order to examine the intrinsic effect of ξ on the nature of the differential operators in equations (22), (23) and (24), we first note that, even excluding the perpendicular part of the Laplacian in $\mathscr{D}$, i.e. $\mathbf{n} \wedge \nabla . \mathbf{n} \wedge \nabla$ in expression (22), the remaining parts of the differential operator in the equations for parallel and perpendicular components are different. Obtaining $\mathbf{n}\,.\,\bar{\nabla}$ from equation (9) and using expression (22), we have

$$\mathscr{D} \equiv \mathbf{n} \wedge \nabla\,.\,\mathbf{n} \wedge \nabla + \frac{1}{\xi}\left\{\left(\mathbf{n}\,.\,\nabla + \frac{1-\xi}{v}\frac{\partial}{\partial t}\right)^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2}\right\} \tag{25}$$

and the operator in equations (23) and (24) is

$$(\mathbf{n}\,.\,\bar{\nabla})^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2} \equiv \left(\mathbf{n}\,.\,\nabla + \frac{1-\xi}{v}\frac{\partial}{\partial t}\right)^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2}. \tag{26}$$

In order to express the operators in the normal form, let us introduce the transformation

$$t' = t + (\xi - 1)\frac{\mathbf{n}\,.\,\mathbf{r}}{v}, \qquad \mathbf{r}' = \mathbf{r} \tag{27}$$

so that

$$\left.\begin{aligned} \mathbf{n}\,.\,\nabla + \frac{1-\xi}{v}\frac{\partial}{\partial t} &\equiv \mathbf{n}\,.\,\nabla' \\ \text{and} \qquad \frac{\partial}{\partial t} &\equiv \frac{\partial}{\partial t'} \end{aligned}\right\}. \tag{28}$$

It may be noted that this transformation is analogous to the Galilean transformation with the role of space and time interchanged. They have been studied by the author (Sen Gupta 1966) as another limit of the Lorentz transformation. With these variables

$$\mathscr{D} \equiv \mathbf{n} \wedge \nabla'\,.\,\mathbf{n} \wedge \nabla' + \frac{1}{\xi}\left\{(\mathbf{n}\,.\,\nabla')^2 - \frac{\xi^2}{u^2}\frac{\partial^2}{\partial t'^2}\right\} \tag{29}$$

and

$$(\mathbf{n}\,.\,\bar{\nabla})^2-\frac{\xi^2}{u^2}\frac{\partial^2}{\partial t^2}\equiv(\mathbf{n}\,.\,\nabla')^2-\frac{\xi^2}{u^2}\frac{\partial^2}{\partial t'^2}. \tag{30}$$

We first note that the differential operators are hyperbolic in character whatever the value of ξ. But the physical properties of the field quantities, for example the evolution in time and the propagation in space, associated with equations (20), (21), (23) and (24), change conspicuously with change in sign of ξ. It is often vaguely mentioned in the literature that the field equations change from hyperbolic to elliptic when ξ changes in sign from positive to negative, i.e. v increases from $v < u$ to $v > u$. This is not true in general, as is evident from above.

(i) *Case* $v < u, \xi > 1$

In the primed system the signature of the space–time associated with the operators (29) and (30) is the usual one, namely $(1, 1, 1, -1)$. The only change effected in the transition to the original system, by the inverse transformation

$$t = t'-(\xi-1)\frac{\mathbf{n}\,.\,\mathbf{r}'}{v}, \qquad \mathbf{r} = \mathbf{r}' \tag{31}$$

is the introduction of a relative retarded or advanced time, depending on the position along the parallel direction. This evidently does not change any physical character of the field quantities.

(ii) *Case* $v > u, \xi < 0$

In the primed system, although the operator (29) is still hyperbolic, the signature of the space–time corresponding to the operator (29) is now $(1, 1, -1, 1)$. This interchanges the role of time with the component of space directed along the velocity. Thus the physical properties associated with the field are expected to change basically. Further, the transition to the original system according to equation (31) is to introduce a relative advanced or retarded time, i.e. in an order opposite to the previous case, because $\xi-1$ is now negative. In fact, with respect to a moving system the concept of retarded or advanced potentials or fields in general becomes relative if a velocity greater than the phase velocity is allowed.

Next, the change in sign of ξ does not change the character of the operator (30), i.e. of equations (23) and (24); but it should be noted that the presence of $\mathbf{n}.\mathbf{H}$ and $\mathbf{n}.\mathbf{E}$ on the right-hand side of equations (23) and (24) effectively changes the nature of the perpendicular components.

Another interesting aspect of the operators (29) and (30) which may be noted is that, with sources which are independent of $\mathbf{n}\wedge\mathbf{r}$, the perpendicular part of the Laplacian being absent, both operators remain hyperbolic for all values of ξ. From equations (20) and (21) the only change in the problem is the change in sign of the terms containing the current with change in sign of ξ.

Finally, after determining the nature of the differential equations (20), (21) and (23), (24), we find that they do not present any further complications for the solution. It is not irrelevant to mention that the effect of the medium in the above discussions is only through ξ, which depends on the velocity only for material media; $\xi = 1$ in empty space.

2.1. *The Poynting theorem*

In the usual manner one can obtain from the field equations (13)–(16)

$$\bar{\nabla}\,.\,\mathbf{E}\wedge\mathbf{H}+\frac{1}{c}\frac{\partial}{\partial t}\Big\{\tfrac{1}{2}\xi\epsilon(\mathbf{n}\wedge\mathbf{E}\,.\,\mathbf{n}\wedge\mathbf{E})$$

$$+\tfrac{1}{2}\xi\mu(\mathbf{n}\wedge\mathbf{H}\,.\,\mathbf{n}\wedge\mathbf{H})+\tfrac{1}{2}\epsilon(\mathbf{n}\,.\,\mathbf{E})^2+\tfrac{1}{2}\mu(\mathbf{n}\,.\,\mathbf{H})\Big\} = -\mathbf{E}\,.\,\mathbf{J}. \tag{32}$$

This can be written in the usual form for the Poynting theorem by substituting for $\bar{\nabla}$ from equation (9):

$$\nabla \cdot \mathbf{E} \wedge \mathbf{H} + \frac{1}{c}\frac{\partial}{\partial t}\Big[\xi \{\tfrac{1}{2}\epsilon(\mathbf{n} \wedge \mathbf{E} \cdot \mathbf{n} \wedge \mathbf{E}) + \tfrac{1}{2}\mu(\mathbf{n} \wedge \mathbf{H} \cdot \mathbf{n} \wedge \mathbf{H})\}$$

$$+\tfrac{1}{2}\epsilon(\mathbf{n} \cdot \mathbf{E})^2 + \tfrac{1}{2}\mu(\mathbf{n} \cdot \mathbf{H})^2 + \frac{1-\xi}{\beta}(\mathbf{n} \cdot \mathbf{E} \wedge \mathbf{H})\Big] = -\mathbf{E} \cdot \mathbf{J}. \tag{33}$$

The common expression for energy, the expression in brackets, is no longer positive definite. The square of the parallel components of the field is the same. But the square of the perpendicular components of the field now appears with a factor ξ, which changes sign if $v > u$. Again, the sign of the last term changes owing to the factor $1-\xi$, which is negative for $v < u$ and positive for $v > u$. Beyond these observations on the difference in the nature of the problem in the cases of $v < u$ and $v > u$, no definite statement may be made for equation (33) without a knowledge of the specific nature of $\mathbf{E}$ and $\mathbf{H}$.

In the investigations of the Čerenkov radiation one is interested in the special cases of P, $\mathbf{J}$, and hence one loses sight of some of these general properties of the field satisfying Maxwell's equations and Minkowski's relations. The equations (20)–(24) show clearly that, even in a system with steady charge current, i.e. $\partial P/\partial t = 0$, $\partial \mathbf{J}/\partial t = 0$, the field produced is not necessarily steady. The time variation enters through the material equation and under suitable conditions the system may radiate.

3. An alternative approach

In the case of a homogeneous medium the study of the electromagnetic field due to given charges and currents may be very conveniently reduced to problems which are mathematically equivalent to those of empty space. Hence it would show clearly radiative and other physical properties of the system. In order to show this, we start with Maxwell's equation in the system at rest in the medium. Let us introduce

$$\mathbf{e} = \sqrt{\epsilon}\mathbf{E}, \qquad \mathbf{h} = \sqrt{\mu}\mathbf{H} \tag{34}$$

$$\mathbf{j} = \sqrt{\mu}\mathbf{J} \quad \text{and} \quad \rho = P/\sqrt{\epsilon}. \tag{35}$$

Maxwell's equations with these variables are

$$\nabla \wedge \mathbf{h} - \frac{1}{u}\frac{\partial \mathbf{e}}{\partial t} = \mathbf{j} \tag{36}$$

$$\nabla \wedge \mathbf{e} + \frac{1}{u}\frac{\partial \mathbf{h}}{\partial t} = 0 \tag{37}$$

$$\nabla \cdot \mathbf{e} = \rho \tag{38}$$

$$\nabla \cdot \mathbf{h} = 0. \tag{39}$$

The above equations are exactly in the same form as those in empty space, with c replaced by u. Hence all the invariance properties of Maxwell's equations in empty space are valid *mutatis mutandis*, with c replaced by u. Thus they are invariant with respect to the linear transformation, similar to the Lorentz transformation with velocity v and c replaced by u.

(i) *Case $v < u$*

The equations (36)–(39) are invariant with respect to the linear transformation:

$$\mathbf{n} \cdot \mathbf{r}' = (\mathbf{n} \cdot \mathbf{r} - vt)\gamma, \qquad t' = \left(t - \frac{v}{u^2}\mathbf{n} \cdot \mathbf{r}\right)\gamma, \qquad \mathbf{n} \wedge \mathbf{r}' = \mathbf{n} \wedge \mathbf{r} \tag{40}$$

$$\rho' = \left(\rho - \frac{\mathbf{v} \cdot \mathbf{j}}{u}\right)\gamma, \qquad (\mathbf{j}' \cdot \mathbf{n}) = \left(\mathbf{j} \cdot \mathbf{n} - \frac{v}{u}\rho\right)\gamma, \qquad \mathbf{n} \wedge \mathbf{j}' = \mathbf{n} \wedge \mathbf{j} \tag{41}$$

and

$$\mathbf{n}\,.\,\mathbf{e}' = \mathbf{n}\,.\,\mathbf{e}, \qquad \mathbf{n}\wedge\mathbf{e}' = \left\{\mathbf{n}\wedge\mathbf{e}+\frac{1}{u}\mathbf{v}\wedge(\mathbf{n}\wedge\mathbf{h})\right\}\gamma$$

$$\mathbf{n}\,.\,\mathbf{h}' = \mathbf{n}\,.\,\mathbf{h}, \qquad \mathbf{n}\wedge\mathbf{h}' = \left\{\mathbf{n}\wedge\mathbf{h}-\frac{1}{u}\mathbf{v}\wedge(\mathbf{n}\wedge\mathbf{e})\right\}\gamma \tag{42}$$

where

$$\gamma = +\left(1-\frac{v^2}{u^2}\right)^{-1/2}. \tag{43}$$

These transformations are exactly the same as the Lorentz transformations, except that c is replaced by u. They are meaningful as linear space–time transformations and, as in the Lorentz transformations, $\mathbf{r}\,.\,\mathbf{r}-u^2t^2$ is invariant. The invariance property is maintained as long as the charge and current are taken as external entities. They have no physical significance, but they are merely mathematical aids. The invariance property is lost as soon as the particle equation is introduced. Problems, in which charges and currents are of the form $f(\mathbf{r}-\mathbf{v}t)$, are reduced to the static case by this transformation. Since the static case is easy to handle, it is convenient to find $\mathbf{e}'$ and $\mathbf{h}'$, from which one can obtain $\mathbf{E}$ and $\mathbf{H}$ by the inverse transformations. Now since the transformations are linear, the physical nature of the field, for example, radiating or not, are the same in both systems. An obvious example is the motion of a charged particle in a homogeneous medium. By this transformation the problem is reduced to the analogous case of static charge in empty space as long as $v < u$, and hence there cannot be any radiation.

(ii) *Case* $v > u$

In this case also one can introduce the transformation

$$\mathbf{n}\,.\,\mathbf{r}' = \left(\mathbf{n}\,.\,\mathbf{r}-\frac{u^2}{v}t\right)\bar{\gamma}, \qquad t' = \left(t-\frac{\mathbf{n}\,.\,\mathbf{r}}{v}\right)\bar{\gamma}, \qquad \mathbf{n}\wedge\mathbf{r}' = \mathbf{n}\wedge\mathbf{r} \tag{44}$$

$$\rho' = \left\{\rho-\frac{u}{v}(\mathbf{n}\,.\,\mathbf{j})\right\}\bar{\gamma}, \qquad \mathbf{n}\,.\,\mathbf{j}' = \left(\mathbf{n}\,.\,\mathbf{j}-\frac{u}{v}\rho\right)\bar{\gamma}, \qquad \mathbf{n}\wedge\mathbf{j}' = \mathbf{n}\wedge\mathbf{j} \tag{45}$$

and

$$\mathbf{n}\,.\,\mathbf{e}' = \mathbf{n}\,.\,\mathbf{e}, \qquad \mathbf{n}\wedge\mathbf{e}' = \left\{\mathbf{n}\wedge\mathbf{e}+\frac{u}{v^2}\mathbf{v}\wedge(\mathbf{n}\wedge\mathbf{h})\right\}\bar{\gamma} \tag{46}$$

$$\mathbf{n}\,.\,\mathbf{h}' = \mathbf{n}\,.\,\mathbf{h}, \qquad \mathbf{n}\wedge\mathbf{h}' = \left\{\mathbf{n}\wedge\mathbf{h}-\frac{u}{v^2}\mathbf{v}\wedge(\mathbf{n}\wedge\mathbf{e})\right\}\bar{\gamma} \tag{47}$$

where

$$\bar{\gamma} = \left(1-\frac{u^2}{v^2}\right)^{-1/2}. \tag{48}$$

This transformation is obtained from the previous one on replacing v by u^2/v. The equations (36)–(39) are also invariant with respect to this transformation. They have no direct similarity with the Lorentz transformation. As before, they have no physical significance, but they also maintain $\mathbf{r}.\mathbf{r}-u^2t^2$ invariant. They have been studied by the author (Sen Gupta 1966) in a different context. By this transformation, it may be possible to reduce a given charge current to some simple form for which the field quantities are easy to determine. Finally, one can obtain $\mathbf{E}$ and $\mathbf{H}$ by the inverse transformation. Again, since the transformations are linear the physical nature of the field remains the same. An important application of this is the Čerenkov effect, which is treated in the following section.

4. The Čerenkov radiation

For a particle moving with velocity $v > u$ along the z direction

$$P = q\delta(x)\delta(y)\delta(z-vt), \qquad \mathbf{J} = \frac{\mathbf{v}}{c}P. \tag{49}$$

According to equations (44) and (48), the transformed charge currents are

$$\rho = 0, \qquad \mathbf{j} = \bar{\gamma}\sqrt{\mu}\mathbf{J}. \tag{50}$$

The field equations are

$$\nabla \wedge \mathbf{h} - \frac{1}{u}\frac{\partial \mathbf{e}}{\partial t} = \frac{\mathbf{v}q}{u\sqrt{\epsilon}}\delta(x)\delta(y)\delta(tv) \tag{51}$$

$$\nabla \wedge \mathbf{e} + \frac{1}{u}\frac{\partial \mathbf{h}}{\partial t} = 0 \tag{52}$$

$$\nabla \cdot \mathbf{e} = 0 \tag{53}$$

and

$$\nabla \cdot \mathbf{h} = 0. \tag{54}$$

In equations (50)–(54) primes are dropped. Thus the problem reduces to an analogous problem of antenna. Since the current is proportional to $\mathbf{v}\delta(x)\delta(y)\delta(t)$, the antenna is a transient one and it is extended infinitely along the z axis. If one considers the Fourier resolution of the current, the strengths for different frequencies are the same and constant.

We next proceed in exactly the same manner as in an antenna problem. Since $\rho = 0$, the field may be described by the vector potential $\mathbf{a}$, such that

$$\mathbf{h} = \nabla \wedge \mathbf{a}, \qquad \mathbf{e} = -\frac{1}{u}\frac{\partial \mathbf{a}}{\partial t} \tag{55}$$

and

$$\nabla \cdot \mathbf{a} = 0.$$

The equation for $\mathbf{a}$ is

$$\left(\nabla \cdot \nabla - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{a} = -\frac{\mathbf{v}q}{\sqrt{\epsilon}u}\delta(x)\delta(y)\delta(tv). \tag{56}$$

This reduces to a two-dimensional wave equation as z does not appear on the right-hand side, and we are interested in the solution $\mathbf{a} \to 0$ as $z \to \pm\infty$. Hence

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{a} = -\frac{v}{\sqrt{\epsilon}u}q\delta(x)\delta(y)\delta(tv) \tag{57}$$

the solution of which is well known; for the given boundary condition it is

$$\mathbf{a} = \frac{\mathbf{n}q}{2\pi\sqrt{\epsilon}}\int_0^t \left[\iint_{S<u(t-\tau)} \frac{\delta(\zeta)\delta(\eta)\,d\zeta\,d\eta}{\{u^2(t-\tau)^2 - S^2\}^{1/2}}\right]\delta(\tau)\,d\tau \tag{58}$$

where $S = +\{(x-\zeta)^2 - (y-\eta)^2\}^{1/2}$, so that

$$\mathbf{a} = \frac{\mathbf{n}q}{2\pi\sqrt{\epsilon}}\frac{1}{(u^2t^2 - x^2 - y^2)^{1/2}} \tag{59}$$

for $ut > +(x^2+y^2)^{1/2}$, and zero otherwise. Finally, on returning to the original system the vector and scalar potentials are given by

$$\Phi = \frac{q}{2\pi\epsilon}\frac{1}{[(z-vt)^2 - \{(v^2-u^2)/u^2\}(x^2+y^2)]^{1/2}} \tag{60}$$

and

$$\mathbf{A} = \mathbf{n}\left(\frac{c}{u}\right)^2 \beta\Phi \tag{61}$$

for $u(vt-z) > \{(v^2-u^2)(x^2+y^2)\}^{1/2}$, and zero otherwise. Thus we obtain the well-known result. Since this case reduces to a problem analogous to that of an antenna, it must radiate. It may be mentioned that the criterion for the radiation, namely the flow of energy momentum across a closed surface, remains the same, as a closed surface remains closed by the linear transformation for a fixed instant of time. We can also find the expression for radiative power per unit frequency interval. For this purpose we start with the Fourier resolutions $\mathbf{e}_\omega$ and $\mathbf{h}_\omega$ of the field intensities; one can easily obtain them directly from the field equations (51)–(53) as $\partial/\partial z \equiv 0$. The surface integral of the corresponding Poynting vector taken along $x^2+y^2 = \text{const.}$ is

$$\oint (\mathbf{e}_\omega \wedge \mathbf{h}_\omega \, . \, \delta\boldsymbol{\sigma}) \exp\{i(\omega+\omega')t\}\, d\omega\, d\omega'\, dz. \tag{62}$$

Now, in order to ensure that the surface is a closed one at a fixed instant of time in the original system, we must have

$$dz + v\, dt = 0 \tag{63}$$

so that we can replace dz in the expression (62) by $-v\, dt$ and perform the integration. Thus one can also find the expression for the energy loss per unit time. Since the expression is well known we shall not repeat the calculations here.

Appendix

In the case of $v = u$, i.e. $1-\epsilon\mu\beta^2 = 0$, Minkowski's material relations are no longer independent; hence **D** and **B** cannot be uniquely determined from them. But the relations along the parallel direction, namely equations (7′) and (8′), are still valid. Minkowski's relations only give

$$\mathbf{D} - \epsilon\beta\mathbf{n} \wedge \mathbf{B} = \epsilon\mathbf{E} - \beta\mathbf{n} \wedge \mathbf{H} \tag{A1}$$

and, from equations (7) and (8),

$$\left.\begin{aligned} \mathbf{n} \wedge \mathbf{H} &= \epsilon\beta\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{E}) \\ \mathbf{n} \wedge \mathbf{E} &= -\mu\beta\mathbf{n} \wedge (\mathbf{n} \wedge \mathbf{H}) \end{aligned}\right\}. \tag{A2}$$

Thus **E** and **H** are always orthogonal to each other. From equation (A1) and Maxwell's equations (1) and (2)

$$\nabla \wedge \mathbf{H} + \epsilon\beta\mathbf{n} \wedge (\nabla \wedge \mathbf{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\epsilon\mathbf{E} - \beta\mathbf{n} \wedge \mathbf{H}) = \mathbf{J}. \tag{A3}$$

E and **H** can be determined from equations (A2) and (A3). As before, the equations for the parallel and perpendicular components are different. Without going into the details of the calculations we give the wave equations of the field intensities. For the parallel components

$$\mathscr{D}'(\mathbf{n}\, .\, \mathbf{E}) = \frac{1}{\epsilon\beta}\left\{\nabla . \mathbf{J} + \left(\mathbf{n}\, .\, \nabla + \frac{1+\beta^2}{u}\frac{\partial}{\partial t}\right)(\mathbf{n} . \mathbf{J})\right\} \tag{A4}$$

$$\mathscr{D}'(\mathbf{n}\, .\, \mathbf{H}) = \nabla\, .\, \mathbf{n} \wedge \mathbf{J} \tag{A5}$$

where

$$\mathscr{D}' \equiv \mathbf{n} \wedge \nabla\, .\, \mathbf{n} \wedge \nabla - \frac{2}{u}(\mathbf{n}\, .\, \nabla)\frac{\partial}{\partial t} - \frac{(1+\beta^2)}{u^2}\frac{\partial^2}{\partial t^2}. \tag{A6}$$

Next, for the perpendicular components,

$$\epsilon\beta\left\{2(\mathbf{n}\, .\, \nabla) + \frac{1+\beta^2}{u}\frac{\partial}{\partial t}\right\}\mathbf{n} \wedge \mathbf{E} = \mathbf{n} \wedge \{\nabla \wedge \mathbf{n}(\mathbf{n}\, .\, \mathbf{H}) + \epsilon\beta\nabla(\mathbf{n}\, .\, \mathbf{E}) - \mathbf{J}\} \tag{A7}$$

$$\left\{2(\mathbf{n}\, .\, \nabla) + \frac{1+\beta^2}{u}\frac{\partial}{\partial t}\right\}\mathbf{n} \wedge \mathbf{H} = \mathbf{n} \wedge [\nabla(\mathbf{n}\, .\, \mathbf{H}) + \mathbf{n} \wedge \{\nabla(\mathbf{n}\, .\, \mathbf{E}) - \mathbf{J}\}]. \tag{A8}$$

There are some important points to be noted in the above equations for **E** and **H**. In the first place, the charge ρ does not appear in the above equations; this is obvious as Maxwell's equations (3) and (4) are not used. Hence ρ does not contribute to producing the field. (3) and (4) may have some role in determining the boundary condition for the field quantities and also for determining **D** from equation (3). Next, the equations for the pendicular components are linear in space and time derivatives, so that with a knowledge of the parallel components they are reduced to quadratures. It is sufficient to know only their value at a given region in space–time to determine them uniquely.

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